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Fixed Point Theorem For Four Weakly Compatible Mappings in Non Archimedean Menger PM-Spaces

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Abstract— In the present paper we prove a unique common fixed point theorem for four weakly compatible self maps in non Archimedean Menger Probabilistic Metric spaces without using the notion of continuity. Our result generalizes and extends the results of Amit Singh, R.C. Dimri and Sandeep Bhatt [A common fixed point theorem for weakly compatible mappings in non-Archimedean Menger PM-space, MATEMATIQKI VESNIK 63, 4 (2011), 285-294] and Khan and Sumitra [A common fixed point theorem in non-Archimedean Menger PM-space, Novi Sad J. Math. 39 (1) (2009), 81-87] and others.

Index Terms— Common fixed points, Compatible mappings, Non-Archimedean Menger PM-spaces, Four self maps.

1 Introduction

Non-Archimedean probabilistic metric spaces and some topological preliminaries on them were first studied by Istrătescu and Crivăt [9] (see also [8]). Some fixed point theorems for mappings on non-Archimedean Menger spaces have been proved by Istrătescu [6, 7] as a result of the generalizations of some of the results of Sehgal and Bharucha-Reid [16] and Sherwood [17]. Achari [1] studied the fixed points of quasi-contraction type mappings in non-Archimedean PM-spaces and generalized the results of Istrătescu [7]. Khan and Sumitra [13] proved a common fixed point theorem for three pointwise R-weakly commuting mappings in complete non-Archimedean Menger PM-spaces. Recently Amit Singh, R.C. Dimri and Sandeep Bhatt prove a unique common fixed point theorem for four weakly compatible self maps in non-Archimedean Menger PM-spaces without using the notion of continuity. Our result generalizes and extends the results of Amit Singh, R.C. Dimri and Sandeep Bhatt [19] and others.

2 Preliminaries

Definition 2.1. [7,9] Let $\boldsymbol{X}$ be any non-empty set and $\boldsymbol{D}$ be the set of all left continuous distribution functions. An ordered pair $(\boldsymbol{X},\boldsymbol{F})$ is said to be non-Archimedean probabilistic metric space (*N.A. PM*-space) if $\boldsymbol{F}$ is a mapping from $\boldsymbol{X}\times\boldsymbol{X}$ into $\boldsymbol{D}$ satisfying the following conditions, where the value of $\boldsymbol{F}$ at $(\boldsymbol{x},\boldsymbol{y})\in\boldsymbol{X}\times\boldsymbol{X}$ is represented by $\boldsymbol{F_{x,y}}$ or $\boldsymbol{F(x,y)}$ for all $\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{X}$ such that

i. $F(x,y;t)=1$ for all $t>0$ if only if $x=y$;
ii. $F(x,y;t)=F(y,x;t)$;
iii. $F(x,y;0)=0$;
iv. If $F(x,y;t_1)=F(y,z;t_2)=1$ then $F(x,z;max\{t_1,t_2\})=1$ for all $x,y,z\in X$.

Definition 2.2. [14] A $\boldsymbol{t}$-norm is a function $\Delta:[\mathbf{0},\mathbf{1}]\times[\mathbf{0},\mathbf{1}]\to[\mathbf{0},\mathbf{1}]$ which is associative, commutative, non-decreasing in each coordinate and $\Delta(\boldsymbol{a},\mathbf{1})=\boldsymbol{a}$ for all $\boldsymbol{a}\in[\mathbf{0},\mathbf{1}]$.

Definition 2.3. [8,10] A non-Archimedean Menger *PM*-space is an ordered triplet (X,F,Δ), where Δ is a t-norm and (X,F) is an *N.A. PM*-space satisfying the following condition:

$F(x,z;max\{t_1,t_2\})\geq\Delta\big(F(x,y;t_1),F(y,z;t_2)\big)$ for all $x,y,z\in X, t_1,t_2\geq 0$.

For details of topological preliminaries on non-Archimedean Menger *PM*-spaces, we refer to Cho, Ha and Chang [3].

Definition 2.4. [2,3] An *N.A.* Menger *PM*-space (X,F,Δ) is said to be of type $(C)g$ if there exists a $g\in\Omega$ such that $g\big(F(x,z;t)\big)\leq g\big(F(x,y;t)\big)+g\big(F(y,z;t)\big)$ for all $x,y,z\in X, t\geq 0$, where $\Omega=\{g\mid g:[0,1]\to[0,1)$ is continuous, strictly decreasing with $g(1)=0$ and $g(0)<\infty\}$.

Definition 2.5. [2,3] An *N.A.* Menger *PM*-space (X,F,Δ) is said to be of type $(D)g$ if there exists a $g\in\Omega$ such that $g\big(\Delta(t_1,t_2)\big)\leq g(t_1)+g(t_2)$ for all $t_1,t_2\in[0,1]$.

Remark 2.1. [2,3] (i) If *N.A.* Menger *PM*-space is of type $(D)g$ then (X,F,Δ) is of type $(C)g$.
(ii) If (X,F,Δ) is *N.A.* Menger *PM*-space and $\Delta\geq\Delta(r,s)=max(r+s-1,1)$, then (X,F,Δ) is of type $(D)g$ for $g\in\Omega$ and $g(t)=1-t$.

Throughout this paper (X,F,Δ) is a complete *N.A.* Menger *PM*-space with a continuous strictly increasing t-norm Δ.

Let $\phi:[0,\infty)\to[0,\infty)$ be a function satisfying the condition

ϕ is upper semi-continuous from the right and $\phi(t) < t$ for $t > 0$.

Definition 2.6. [2,3] A sequence $\{x_n\}$ in the *N.A.* Menger *PM*-space (X, F, Δ) converges to x if and only if for each $\epsilon > 0, \lambda > 0$ there exists $M(\epsilon, \lambda)$ such that $g\big(F(x_n, x; \epsilon)\big) < g(1-\lambda)$ for all $n > M$.

Definition 2.7. [3] A sequence $\{x_n\}$ in the *N.A.* Menger *PM*-space is a Cauchy sequence if and only if for each $\epsilon > 0, \lambda > 0$ there exists $M(\epsilon, \lambda)$ such that $g\big(F(x_n, x_n; \epsilon)\big) < g(1-\lambda)$ for all $n > M$ and $p \geq 1$.

Example 2.1. [3] Let X be any se with at least two elements. If we define $F(x, x; t) = 1$ for all $x \in X, t > 0$ and $F(x, y; t) = \{0 \; if \; t \leq 1 \; and \; 1 \; if \; t > 1\}$, where $x, y \in X, x \neq y$, then (X, F, Δ) is the *N.A.* Menger *PM*-space with $\Delta(a, b) = min(a, b)$ or (a, b).

Example 2.2. [3] Let $X = R$ be the set of real numbers equipped with metric defined as $d(x, y) = |x - y|$. Set $F(x, y; t) = \frac{t}{t + d(x,y)}$. Then (X, F, Δ) is an *N.A.* Menger *PM*-space with Δ as a continuous t-norm $\Delta(r, s) = min(r, s)$ or (r, s).

Lemma 2.1. [3] If a function $\phi: [0, \infty) \to [0, \infty)$ satisfies the condition (Φ), then we get

(i) for all $t \geq 0, \lim_{n\to\infty} \phi^n(t) = 0$, where $\phi^n(t)$ is the n^{th} iteration of $\phi(t)$,

(ii) if $\{t_n\}$ is a non-decreasing sequence of real numbers and $t_{n+1} \leq \phi(t_n), n = 1,2, \ldots,$ then $\lim_{n\to\infty} t_n = 0$. In particular, if $t \leq \phi(t)$, for each $t \geq 0$, then $t = 0$.

Lemma 2.2. [3] Let $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} F(y_n, y_{n+1}, t) = 1$ for each $t > 0$. If $\{y_n\}$ is not a Cauchy sequence in X, then there exist $\epsilon_0 > 0, t_0 > 0$, and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that

(i) $m_i > n_{i+1}$ and $n_i \to \infty$ as $i \to \infty$.

(ii) $F(y_{mi}, y_{mi}; t_0) < 1 - \epsilon_0$ and $F(y_{mi-1}, y_{ni}; t_0) \geq 1 - \epsilon_0, i = 1,2, \ldots$

Definition 2.8. [10] Let $A, S: X \to X$ be mappings. A and S are said to be compatible if $\lim_{n\to\infty} g\big(F(AS_{xn}, SA_{xn}, t)\big) = 0$ for all $t > 0$, when $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = z = \lim_{n\to\infty} Sx_n$ for some $z \in X$.

Definition 2.9. [11,12] Let $A, S: X \to X$ be mappings. A and S are said to be compatible if they commute at coincidence points. That is, if $Ax = Sx$ implies that $ASx = SAx$, for x in X.

3 Main Results

Theorem 3.1. Let (X, F, Δ) be a complete *N.A.* Menger *PM*-space and $A, B, S, T: X \to X$ be mappings satisfying

(i) $A(X) \subseteq T(X), B(X) \subseteq S(X)$,

(ii) The pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible and

(iii) $g(F(Ax, By; t)) \leq \phi[\max\{g\big(F(Sx, Ty; t)\big), g\big(F(Sx, Ax; t)\big), g\big(F(Ty, By; t)\big), \frac{1}{2}\big(g\big(F(Sx, By; t)\big) + g\big(F(Ty, Ax; t)\big)\big), \frac{1}{2}\Big(g\big(F(Sx, Ax; t)\big) + g\big(F(Ty, By; t)\big)\Big)\}]$

for every $x, y \in X$, where ϕ satisfies the condition Φ. Then A, B, S and T have a unique common fixed point in X.

Proof: Since $A(X) \subseteq T(X)$, for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subseteq S(X)$, for this x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$ and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \text{ for } n = 1,2, \ldots \quad (1)$$

Let $M_n = g\big(F(Ax_n, Bx_{n+1}; t)\big) = g\big(F(y_n, y_{n+1}; t)\big)$ for $n = 1,2, \ldots$ Then,

$$\begin{aligned}
M_{2n} &= g\big(F(Ax_{2n}, Bx_{2n+1}; t)\big) \\
&\leq \phi[\max\{g\big(F(Sx_{2n}, Tx_{2n+1}; t)\big), g\big(F(Sx_{2n}, Ax_{2n}; t)\big), g\big(F(Tx_{2n+1}, Bx_{2n+1}; t)\big), \\
&\quad \tfrac{1}{2}\Big(g\big(F(Sx_{2n}, Bx_{2n+1}; t)\big) + g\big(F(Tx_{2n+1}, Ax_{2n}; t)\big)\Big), \\
&\quad \tfrac{1}{2}\Big(g\big(F(Sx_{2n}, Ax_{2n}; t)\big) + g\big(F(Tx_{2n+1}, Bx_{2n+1}; t)\big)\Big)\}] \\
&\leq \phi[\max\{g\big(F(y_{2n-1}, y_{2n}; t)\big), g\big(F(y_{2n-1}, y_{2n}; t)\big), g\big(F(y_{2n}, y_{2n+1}; t)\big), \\
&\quad \tfrac{1}{2}\Big(g\big(F(y_{2n-1}, y_{2n+1}; t)\big) + g\big(F(y_{2n}, y_{2n}; t)\big)\Big), \\
&\quad \tfrac{1}{2}\Big(g\big(F(y_{2n-1}, y_{2n}; t)\big) + g\big(F(y_{2n}, y_{2n+1}; t)\big)\Big)\}] \\
&\leq \phi[\max\{g\big(F(y_{2n-1}, y_{2n}; t)\big), g\big(F(y_{2n-1}, y_{2n}; t)\big), g\big(F(y_{2n}, y_{2n+1}; t)\big), \\
&\quad \tfrac{1}{2}\Big(g\big(F(y_{2n-1}, y_{2n}; t)\big) + g\big(F(y_{2n}, y_{2n+1}; t)\big)\Big), \\
&\quad \tfrac{1}{2}\Big(g\big(F(y_{2n-1}, y_{2n}; t)\big) + g\big(F(y_{2n}, y_{2n+1}; t)\big)\Big)\}]
\end{aligned}$$

i.e.

$$M_{2n} \leq \phi\Big[\max\Big\{M_{2n-1}, M_{2n-1}, M_{2n}, \tfrac{1}{2}(M_{2n-1}, +M_{2n}), \tfrac{1}{2}(M_{2n-1}, +M_{2n})\Big\}\Big] \quad (2)$$

If $M_{2n} > M_{2n-1}$ then by (2) $M_{2n} \geq \phi(M_{2n})$ a contradiction. If $M_{2n-1} > M_{2n}$ then by (2) $M_{2n} \leq \phi(M_{2n-1})$. So by Lemma 2.1, we have $\lim_{n\to\infty} M_{2n} = 0$, i.e.,

$$\lim_n g\big(F(Ax_{2n}, Bx_{2n+1}; t)\big) = 0 \; i.e. \lim_n g\big(F(y_{2n}, y_{2n+1}; t)\big) = 0$$

Similarly, we can show that

$$\lim_n g\big(F(Bx_{2n+1}, Ax_{2n+2}; t)\big) = 0 \; i.e. \lim_n g\big(F(y_{2n+1}, y_{2n+2}; t)\big) = 0$$

This we have $\lim_n g\big(F(Ax_n, Bx_{n+1}; t)\big) = 0$ for all $t > 0, i.e.$

$$\lim_n g\big(F(y_n, y_{n+1}; t)\big) = 0 \; for \; all \; t > 0 \quad (3)$$

Before proceeding with the proof of the theorem, we first prove the following claim:

CLAIM. Let A, B, S and $T : X \to X$ be maps satisfying (i), (ii) and (iii) and $\{y_n\}$ be defined by (1) such that

$$\lim_n g\big(F(y_n, y_{n+1}; t)\big) = 0 \quad (4)$$

for all n. Then $\{y_n\}$ is a Cauchy sequence.

Proof of Claim. Since $g \in \Omega$, it follows that

$\lim_{n\to\infty} F(y_n, y_{n+1}; t) = 1$ for each $t > 0$ if and only if $\lim_{n\to\infty} g\big(F(y_n, y_{n+1}; t)\big) = 1$ for each $t > 0$.

By Lemma 2.2, if $\{y_n\}$ is not a Cauchy sequence in X, there exists $\epsilon_0 > 0, t_0 > 0$ and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that

(A) $m_i > n_{i+1}$ and $n_i \to \infty$ as $i \to \infty$;

(B) $g\big(F(y_{mi}, y_{ni}; t_0)\big) > g(1-\epsilon_0)$ and $g\big(F(y_{mi-1}, y_{ni}; t_0)\big) \le g(1-\epsilon_0), i = 1,2,\ldots$ Since $g(t) = 1 - t$, we have

$$\begin{aligned} g(1-\epsilon_0) &< g\big(F(y_{mi}, y_{ni}; t_0)\big) \\ &\le g\big(F(y_{mi}, y_{mi-1}; t_0)\big) + g\big(F(y_{mi-1}, y_{ni}; t_0)\big) \\ &\le g\big(F(y_{mi}, y_{mi-1}; t_0)\big) + g(1-\epsilon_0) \end{aligned} \tag{5}$$

As $i \to \infty$ in (5) we have

$$\lim_{n\to\infty} g\big(F(y_{mi}, y_{ni}; t_0)\big) = g(1-\epsilon_0) \tag{6}$$

On the other hand, we have

$$\begin{aligned} g(1-\epsilon_0) &< g\big(F(y_{mi}, y_{ni}; t_0)\big) \\ &\le g\big(F(y_{ni}, y_{ni+1}; t_0)\big) + g\big(F(y_{mi}, y_{ni+1}; t_0)\big) \end{aligned} \tag{7}$$

Now consider $g\big(F(y_{mi-1}, y_{ni+1}; t_0)\big)$ in (7) and assume that both m_i and n_i are even. Then, by (iii), we have

$$\begin{aligned} g\big(F(y_{mi}, y_{ni+1}; t_0)\big) &= g\big(F(Ax_{mi}, Bx_{ni+1}; t_0)\big) \\ &\le \phi\Big[\max\Big\{g\big(F(Sx_{mi}, Tx_{ni+1}; t_0)\big), g\big(F(Sx_{mi}, Ax_{mi}; t_0)\big), \\ &\quad g\big(F(Tx_{ni+1}, Bx_{ni+1}; t_0)\big), \tfrac{1}{2}\Big(g\big(F(Sx_{mi}, Bx_{ni+1}; t_0)\big) \\ &\quad + g\big(F(Tx_{ni+1}, Ax_{mi}; t_0)\big)\Big), \tfrac{1}{2}\Big(g\big(F(Sx_{mi}, Ax_{mi}; t_0)\big) \\ &\quad + g\big(F(Tx_{ni+1}, Bx_{ni+1}; t_0)\big)\Big)\Big\}\Big] \\ &\le \phi\Big[\max\Big\{g\big(F(y_{mi-1}, y_{ni}; t_0)\big), g\big(F(y_{mi-1}, y_{mi}; t_0)\big), \\ &\quad g\big(F(y_{ni}, y_{ni+1}; t_0)\big), \tfrac{1}{2}\Big(g\big(F(y_{mi-1}, y_{ni+1}; t_0)\big) \\ &\quad + g\big(F(y_{ni}, y_{mi}; t_0)\big)\Big), \tfrac{1}{2}\Big(g\big(F(y_{mi-1}, y_{mi}; t_0)\big) \\ &\quad + g\big(F(y_{ni}, y_{ni+1}; t_0)\big)\Big)\Big\}\Big] \end{aligned}$$

Letting $i \to \infty$ in above equation, we have

$$g(1-\epsilon_0) \le \phi[\max\{g(1-\epsilon_0), 0, 0, g(1-\epsilon_0), 0\}]$$

i.e. $g(1-\epsilon_0) \le \phi\big(g(1-\epsilon_0)\big)$, which is a contradiction. Hence the sequence $\{y_n\}$ defined by (1) is a Cauchy sequence, which concludes the proof of the claim.

Since X is complete, then the sequence $\{y_n\}$ converges to a point z in X and so the subsequences $\lim_{n\to\infty} Ax_{2n}$, $\lim_{n\to\infty} Bx_{2n+1}$, $\lim_{n\to\infty} Sx_{2n}$ and $\lim_{n\to\infty} Tx_{2n+1}$ of $\{y_n\}$ also converge to the limit z.

Since $B(X) \subseteq S(X)$, there exists a point $u \in X$ such that $z = Su$. Then, using (iii), we have

$$\begin{aligned} g\big(F(Au, z; t)\big) &\le g\big(F(Au, Bx_{2n-1}; t)\big) + g\big(F(Bx_{2n-1}, z; t)\big) \\ &\le \phi\Big[\max\Big\{g\big(F(Su, Tx_{2n-1}; t)\big), g\big(F(Su, Au; t)\big), g\big(F(Tx_{2n-1}, Bx_{2n-1}; t)\big), \\ &\quad \tfrac{1}{2}\Big(g\big(F(Su, Bx_{2n-1}; t)\big) + g\big(F(Tx_{2n-1}, Au; t)\big)\Big), \tfrac{1}{2}\Big(g\big(F(Su, Au; t)\big) \\ &\quad + g\big(F(Tx_{2n-1}, Bx_{2n-1}; t)\big)\Big)\Big\}\Big] \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} g\big(F(Au, z; t)\big) &\le \phi\Big[\max\Big\{g(z, z; t), g\big(F(z, Au; t)\big), g\big(F(z, z; t)\big), \\ &\quad \tfrac{1}{2}\Big(g\big(F(z, z; t)\big) + g\big(F(z, Au; t)\big)\Big), \\ &\quad \tfrac{1}{2}\Big(g\big(F(z, Au; t)\big) + g\big(F(z, z; t)\big)\Big)\Big\}\Big] \\ &= \phi\Big[\max\Big\{0, g\big(F(z, Au; t)\big), 0, \\ &\quad \tfrac{1}{2}\Big(0 + g\big(F(z, Au; t)\big)\Big), \\ &\quad \tfrac{1}{2}\Big(g\big(F(z, Au; t)\big) + 0\Big)\Big\}\Big] \\ &\le \phi\Big(g\big(F(z, Au; t)\big)\Big) \end{aligned}$$

for all $t > 0$, which implies that $g\big(F(Au, z; t)\big) = 0$ for all $t > 0$ by Lemma 2.1. Therefore $Au = Su = z$. Since $A(X) \subseteq T(X)$, there exists a point v in X such that $z = Tv$. Again using (iii), we have

$$\begin{aligned} g\big(F(z, Bv; t)\big) &= g\big(F(Au, Bv; t)\big) \\ &\le \phi\Big[\max\Big\{g(Su, Tv; t), g\big(F(Su, Au; t)\big), g\big(F(Tv, Bv; t)\big), \\ &\quad \tfrac{1}{2}\Big(g\big(F(Su, Bv; t)\big) + g\big(F(Tu, Au; t)\big)\Big), \\ &\quad \tfrac{1}{2}\Big(g\big(F(Su, Au; t)\big) + g\big(F(Tu, Bv; t)\big)\Big)\Big\}\Big] \\ &\le \phi\Big[\max\Big\{g\big(F(z, z; t)\big), g\big(F(z, z; t)\big), g\big(F(z, Bv; t)\big), \\ &\quad \tfrac{1}{2}\Big(g\big(F(z, Bv; t)\big) + g\big(F(z, z; t)\big)\Big), \\ &\quad \tfrac{1}{2}\Big(g\big(F(z, z; t)\big) + g\big(F(z, Bv; t)\big)\Big)\Big\}\Big] \\ &= \phi\Big[\max\Big\{0, 0, g\big(F(z, Bv; t)\big), \tfrac{1}{2}\Big(g\big(F(z, Bv; t)\big)\Big), \\ &\quad \tfrac{1}{2}\Big(g\big(F(z, Bv; t)\big)\Big)\Big\}\Big] \\ &\le \phi\Big(g\big(F(z, Bv; t)\big)\Big) \text{ for all } t > 0 \end{aligned}$$

which implies that $g\big(F(Bv, z; t)\big) = 0$ for all $t > 0$ by Lemma 2.1. Therefore $Bv = Tv = z$. Since A and S are weakly compatible mappings, $ASz = SAz$ i.e. $Az = Sz$. Now we show that z is a fixed point of A. If $Az \ne z$, then by (iii), we have

$$\begin{aligned} g\big(F(Az, z; t)\big) &= g\big(F(Az, Bv; t)\big) \\ &\le \phi\Big[\max\Big\{g\big(F(Sz, Tv; t)\big), g\big(F(Sz, Az; t)\big), g\big(F(Tv, Bv; t)\big), \\ &\quad \tfrac{1}{2}\Big(g\big(F(Sz, Bv; t)\big) + g\big(F(Tu, Az; t)\big)\Big), \\ &\quad \tfrac{1}{2}\Big(g\big(F(Sz, Az; t)\big) + g\big(F(Tu, Bv; t)\big)\Big)\Big\}\Big] \\ &\le \phi\Big[\max\Big\{g\big(F(Az, z; t)\big), g\big(F(Az, Az; t)\big), g\big(F(z, z; t)\big), \\ &\quad \tfrac{1}{2}\Big(g\big(F(Az, z; t)\big) + g\big(F(z, Az; t)\big)\Big), \\ &\quad \tfrac{1}{2}\Big(g\big(F(Az, Az; t)\big) + g\big(F(z, z; t)\big)\Big)\Big\}\Big] \\ &\le \phi\Big[\max\Big\{g\big(F(Az, z; t)\big), 0, 0, \tfrac{1}{2}\Big(g\big(F(Az, z; t)\big)\Big), \\ &\quad \tfrac{1}{2}\big(g\big(F(z, Az; t)\big), 0\big)\Big\}\Big] \\ &\le \phi\Big(g\big(F(Az, z; t)\big)\Big) \text{ for all } t > 0 \end{aligned}$$

which implies that $g(F(Az,z;t)) = 0$ for all $t > 0$ by Lemma 2.1. Therefore $Az = z$. Hence $AZ = Sz = z$.

Similarly, as B and T are weakly compatible mappings, we have $Bz = Tz = z$, since by (iii), we have

$$\begin{aligned} g(F(z,Bz;t)) &= g(F(Az,Bz;t)) \\ &\le \phi\Big[\max\Big\{g(F(Sz,Tz;t)), g(F(Sz,Az;t)), g(F(Tz,Bz;t)), \\ &\quad \tfrac{1}{2}\big(g(F(Sz,Bz;t)) + g(F(Tz,Az;t))\big), \\ &\quad \tfrac{1}{2}\big(g(F(Sz,Az;t)) + g(F(Tz,Bz;t))\big)\Big\}\Big] \\ &\le \phi\Big[\max\Big\{g(F(z,Bz;t)), 0, 0, \tfrac{1}{2}\big(g(F(z,Bz;t)) \\ &\quad + g(F(Bz,z;t)), 0\big)\Big\}\Big] \\ &\le \phi\big(g(F(Bz,z;t))\big) \text{ for all } t > 0 \end{aligned}$$

which implies that $g(F(Bz,z;t)) = 0$ for all $t > 0$ by Lemma 2.1. Therefore $Bz = z$. Hence $Bz = Tz = z$.

Thus $Az = Bz = Sz = Tz = z$, that is, z is a common fixed point of A, B, S and T.

Finally, in order to prove the uniqueness of z, suppose that w is another common fixed point of A, B, S and T. Then by (iii), we have

$$\begin{aligned} g(F(z,w;t)) &= g(F(Az,Bw;t)) \\ &\le \phi\Big[\max\Big\{g(F(Sz,Tw;t)), g(F(Sz,Az;t)), g(F(Tw,Bw;t)), \\ &\quad \tfrac{1}{2}\big(g(F(Sz,Bw;t)) + g(F(Tw,Az;t))\big), \\ &\quad \tfrac{1}{2}\big(g(F(Sz,Az;t)) + g(F(Tw,Bw;t))\big)\Big\}\Big] \\ &\le \phi\Big[\max\Big\{g(F(z,w;t)), g(F(z,z;t)), g(F(w,w;t)), \\ &\quad \tfrac{1}{2}\big(g(F(z,w;t)) + g(F(w,z;t))\big), \\ &\quad \tfrac{1}{2}\big(g(F(z,z;t)) + g(F(w,w;t))\big)\Big\}\Big] \\ &= \phi\Big[\max\Big\{g(F(z,w;t)), 0, 0, \tfrac{1}{2}\big(g(F(z,w;t)) \\ &\quad + g(F(w,z;t)), 0\big)\Big\}\Big] \\ &\le \phi\big(g(F(z,w;t))\big) \text{ for all } t > 0 \end{aligned}$$

which implies that $g(F(z,w;t)) = 0$ for all $t > 0$ by Lemma 2.1. Hence $z = w$. Therefore z is a unique common fixed point of A, B, S and T.

Corollary 3.1. Let $A, S, T: X \to X$ be the mappings satisfying
(i) $A(X) \subseteq S(X) \cap T(X)$
(ii) the pair $\{A, S\}$ and $\{A, T\}$ are weakly compatible and
(iii) $g(F(Ax,Ay;t)) \le \phi\Big[\max\Big\{g(F(Sx,Ty;t)), g(F(Sx,Ax;t)), g(F(Ty,Ay;t)), \tfrac{1}{2}\big(g(F(Sx,Ay;t)) + g(F(Ty,Ax;t))\big), \tfrac{1}{2}\big(g(F(Sx,Ax;t)) + g(F(Ty,Ay;t))\big)\Big\}\Big]$
for every $x, y \in X$, where ϕ satisfies the condition (Φ). Then A, S and T have a unique common fixed point in X.

Corollary 3.2. Let $A, S : X \to X$ be the mappings satisfying
(i) $A(X) \subseteq S(X)$
(ii) the pair $\{A, S\}$ is weakly compatible and
(iii) $g(F(Ax,Ay;t)) \le \phi\Big[\max\Big\{g(F(Sx,Sy;t)), g(F(Sx,Ax;t)), g(F(Sy,Ay;t)), \tfrac{1}{2}\big(g(F(Sx,Ay;t)) + g(F(Sy,Ax;t))\big), \tfrac{1}{2}\big(g(F(Sx,Ax;t)) + g(F(Sy,Ay;t))\big)\Big\}\Big]$
for every $x, y \in X$, where ϕ satisfies the condition (Φ). Then A and S have a unique common fixed point in X.

We can also derive the following results from Theorem 3.1.

Corollary 3.3. Let S and T be two continuous self maps of a complete *N.A.* Menger *PM*-space (X, F, Δ). Let A be a self-map satisfying
(i) $\{A, S\}$ and $\{A, T\}$ are pointwise R-weakly commuting and $A(X) \subseteq S(X) \cap T(X)$
(ii) $g(F(Ax,Ay;t)) \le \phi[\max\{g(F(Sx,Ty;t)), g(F(Sx,Ax;t)), g(F(Sx,Ay;t)), g(F(Ty,Ay;t))\}]$
for every $x, y \in X$, where ϕ satisfies the condition (Φ). Then A, S and T have a unique common fixed point in X.

Taking $T = S$ in *Corollary 3.3* we get the following corollary unifying Vasukis theorem [2], which in turn also generalizes the result of Pant [15].

Corollary 3.4. Let (X, F, Δ) be a complete *N.A.* Menger *PM*-space and S be a continuous self mapping of X. Let A be another self-mapping of X satisfying that
(i) $\{A, S\}$ is R-weakly commuting and $A(X) \subseteq S(X)$
(ii) $g(F(Ax,Ay,a;t)) \le \phi[\max\{g(F(Sx,Sy;t)), g(F(Sx,Ax;t)), g(F(Sx,Ay;t)), g(F(Sy,Ay;t))\}]$
for every $x, y \in X$, where ϕ satisfies the condition (Φ). Then A and S have a unique common fixed point.

Remark 3.1. In Theorem 3.1, if S and T are continuous and pairs $\{A, S\}$ and $\{B, T\}$ are compatible instead of condition (ii), the theorem remains true.

Remark 3.2. In our generalization the inequality condition (iii) satisfied by the mappings A, B, S and T is stronger than that of Theorem 2 of Khan and Sumitra [13] and Theorem 1.9 of Vasuki [21].

4 An Application

Theorem 4.1. Let (X, F, Δ) be a complete *N.A.* Menger *PM*-space and A, B, S and T be mappings from the product $X \times X$ to X such that

$$\begin{gathered} A(X \times \{y\}) \subseteq T(X \times \{y\}), \qquad B(X \times \{y\}) \subseteq S(X \times \{y\}), \\ g\big(F(A(T(x,y),y), T(A(x,y),y);t)\big) \le g\big(F(A(x,y), T(x,y);t)\big), \\ g\big(F(B(S(x,y),y), S(B(x,y),y);t)\big) \le g\big(F(B(x,y), S(x,y);t)\big) \qquad (8) \end{gathered}$$

for all $t > 0$. If S and T are continuous with respect to their direct argument and

$$g\big(F(A(x,y),B(x',y');t)\big) \leq \phi\Big[\max\Big\{g\big(F(S(x,y),T(x',y');t)\big), g\big(F(S(x,y),A(x,y);t)\big), g\big(F(T(x',y'),B(x',y');t)\big), \tfrac{1}{2}\Big(g\big(F(S(x,y),B(x',y');t)\big) + g\big(F(T(x',y'),A(x,y);t)\big)\Big), \tfrac{1}{2}\Big(g\big(F(S(x,y),A(x,y);t)\big) + g\big(F(T(x',y'),B(x',y');t)\big)\Big)\Big\}\Big] \quad (9)$$

for all $t > 0$ and x, y, x', y' in X, then there exists only one point b in X such that

$$A(b,y) = S(b,y) = B(b,y) = T(b,y)\ \forall\ y \in X$$

Proof: By (8) and (9),

$$g\big(F(A(x,y),B(x',y');t)\big) \leq \phi\Big[\max\Big\{g\big(F(S(x,y),T(x',y');t)\big), g\big(F(S(x,y),A(x,y);t)\big), g\big(F(T(x',y'),B(x',y');t)\big), \tfrac{1}{2}\Big(g\big(F(S(x,y),B(x',y');t)\big) + g\big(F(T(x',y'),A(x,y);t)\big)\Big), \tfrac{1}{2}\Big(g\big(F(S(x,y),A(x,y);t)\big) + g\big(F(T(x',y'),B(x',y');t)\big)\Big)\Big\}\Big]$$

for all $t > 0$, therefore by Theorem 3.1, for each y in X, there exists only one $x(y)$ in X such that

$$A(x(y),y) = S(x(y),y) = B(x(y),y) = T(x(y),y) = x(y),$$

for every y, y' in X and

$$g\big(F(x(y),x(y');t)\big) = g\big(F(A(x(y),y),A(x(y'),y');t)\big) \leq \phi\Big[\max\Big\{g\big(F(A(x,y),A(x',y');t)\big), g\big(F(A(x,y),A(x,y);t)\big), g\big(F(T(x',y'),A(x',y');t)\big), \tfrac{1}{2}\Big(g\big(F(A(x,y),A(x',y');t)\big) + g\big(F(A(x',y'),A(x,y);t)\big)\Big), \tfrac{1}{2}\Big(g\big(F(A(x,y),A(x,y);t)\big) + g\big(F(T(x',y'),A(x',y');t)\big)\Big)\Big\}\Big]$$

This implies that $x(y) = x(y')$ and hence $x(\cdot)$ is some constant $b \in X$ so that

$$A(b,y) = b = T(b,y) = S(b,y) = B(b,y)\ \forall\ y \in X$$

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